

Kaluza-Klein two brane worlds cosmology at low energy

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We study two $(4 + n)$ -dimensional branes embedded in $(5 + n)$ -dimensional spacetime. Using the gradient expansion approximation, we find that the effective theory is described by the $(4 + n)$ -dimensional scalar-tensor gravity with a specific coupling function. Based on this theory we investigate the Kaluza-Klein two brane worlds cosmology at low energy. We study in both the static and the non-static internal dimensions. In the static case the effective gravitational constant in the induced Friedmann equation depends on the equations of state of the brane matters and the dark radiation term naturally appear. In the non-static case we take a relation between the external and internal scale factors of the form $b(t) = a^\gamma(t)$ in which the brane world evolves with two scale factors. In this case, the induced Friedmann equation on the brane is modified in the effective gravitational constant and the term proportional to $a^{-4\beta}$. For dark radiation, we find $\gamma = -2/(1+n)$. Finally, we discuss the issue of conformal frames which naturally arises with scalar-tensor theories. We find that the static internal dimensions in the Jordan frame may become non-static in the Einstein frame.

PACS numbers: 04.50.+h, 98.80.Cq, 98.80.Hw

I. INTRODUCTION

One of the most interesting and surprising aspects of the string theory or M-theory is the fact that it can only be correctly formulated in a higher dimensional spacetime. On the other hand, our observed Universe is a four-dimensional spacetime. Therefore we need a mechanism of compactification of the extra dimensions, so that they become invisible at least at low energy scales. Moreover, investigations of non-perturbative string theory has lead to the discovery that string theory must contain higher dimensional extended objects called branes. The existence of these branes has inspired a new method of compactification of extra dimensions, so that they become invisible at least at low energy scales. Previously the preferred method was Kaluza-Klein compactification, in which the extra dimensions are compact and extremely small. This method of compactification has further inspired a class of classical models of the universe, in which extra dimensions can be included in general relativity, and their possible implications for classical cosmology can be investigated phenomenologically without any dependence on a particular model of string theory. This is

known as the brane world scenario, in which the standard particles or fields are confined to a brane, while the graviton can propagates into the bulk as well as into the brane. Much efforts to reveal cosmology on the brane have been done in the context of five-dimensional spacetime, especially after the stimulating proposals by Randall and Sundrum (RS) [1, 2]. In this model, a five-dimensional realization of the Horava-Witten solution [3], the hierarchy problem can be solved by introducing an appropriated exponential warp factor in the metric. The various properties and characteristics of the RS model have been extensively analyzed: the cosmology framework [4–8], the low energy effective theory [9–20], black hole physics [21–26], the Lorentz violation [27–37], etc. However, the RS model with codimension one brane world is insufficient to reconcile a higher-dimensional theory with the observed four-dimensional spacetime as suggested by string theory.

Recently, the hybrid construction of the Kaluza-Klein and brane world compactifications, i.e., a Kaluza-Klein compactifications on the brane has been investigated [38–45]. Such a way of construction is called Kaluza-Klein brane world. A basic equation for the study of Kaluza-Klein brane worlds in which some dimensions on the brane are compactified or for a regularization scheme for a higher codimension brane world was derived by Yamauchi and Sasaki [43]. To analyzes the Kaluza-Klein cosmology some authors have used the Shiromizu-Maeda-Sasaki equation [4] or solving the bulk geometry. However, it difficult to solve the bulk geometry in most cases.

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In this paper, our main purpose is to study a low energy two brane cosmological models in higher-dimensional spacetime. We generalize the case four-dimensional two brane models to $(4+n)$ -dimensional two brane models where n represents internal dimensions of the brane. We derive the effective equations of motion for higher-dimensional two brane model using a low energy expansion method [13]. This perturbative method solves the full $(5+n)$ -dimensional equations of motion using an approximation and after imposing the junction conditions, one obtains the $(4+n)$ -dimensional effective equations of motion. The effective equations can be solved without knowing the bulk geometry. Based on this theory we discuss the cosmology two brane models at low energy. We study in both the static and the non-static internal dimensions.

This paper is organized as follows. In section II, we study a higher braneworld model in a $(5+n)$ -dimensional spacetime bulk with a cosmological constant. We solve the $(5+n)$ -dimensional Einstein equations at the low energy using the gradient expansion approximation. We see the effective theory is described by the $(4+n)$ -dimensional quasi-scalar-tensor gravity with a specific coupling function. In section III, the Kaluza-Klein two brane worlds cosmology are presented. We derive the effective Friedmann equations both in the static and non-static internal dimensions. Section IV is devoted to the conclusions. In Appendix A, we present detailed calculations.

II. LOW ENERGY EFFECTIVE THEORY FOR HIGHER-DIMENSIONAL TWO BRANE WORLDS

In this section, we derive the low energy effective theory for higher-dimensional two branes system solving the bulk geometry formally in the gradient expansion approximation developed by Kanno and Soda [13] (see also

[12]). We consider that the two branes represent a $(4+n)$ -dimensional spacetime embedded in a $(5+n)$ -dimensional spacetime. We assume that there is no matter in the bulk and the energy-momentum tensor of the bulk is proportional to the $(5+n)$ -dimensional cosmological constant, $-2\Lambda_{5+n} = (4+n)(3+n)/l^2$. Then the higher dimensional braneworld model is described by the action

$$S = \frac{1}{2\kappa^2} \int d^{5+n}x \sqrt{-g} \left[\mathcal{R} + \frac{(4+n)(3+n)}{l^2} \right] - \sum_{i=A,B} \int d^{4+n}x \sqrt{-g^{i\text{brane}}} (\sigma_i - \mathcal{L}_{\text{matter}}^i) , \quad (1)$$

where \mathcal{R} , $g_{\mu\nu}^{i\text{brane}}$, l and κ^2 are the $(5+n)$ -dimensional scalar curvature, the induced metric on branes, the scale of the bulk curvature radius and the gravitational constant in $(5+n)$ -dimensions, respectively. Because we will consider the matter terms in (1), the branes will not in general be flat. Consequently we cannot put both branes at $y = 0$ and $y = l$ and use Gaussian normal coordinates. Therefore, we use the following coordinate system to describe the geometry of the brane model,

$$ds^2 = e^{2\phi(y,x^\mu)} dy^2 + g_{\mu\nu}(y, x^\mu) dx^\mu dx^\nu . \quad (2)$$

The proper distance between A -brane and B -brane with fixed x coordinates can be written as

$$d(x) = \int_0^l e^{\phi(y,x)} dy . \quad (3)$$

The extrinsic curvature is defined as

$$K_{\mu\nu} = -\frac{1}{2} \frac{\partial}{\partial y} g_{\mu\nu} \equiv -\frac{1}{2} g_{\mu\nu,y} . \quad (4)$$

In the coordinate system (2) and using the extrinsic curvature (4), we can write down the components of the Einstein equations in $(5+n)$ -dimensions as

$$\begin{aligned} {}^{(5+n)}G^\mu{}_\nu &= G^\mu{}_\nu + e^{-\phi} (e^{-\phi} K^\mu{}_\nu - \delta^\mu{}_\nu e^{-\phi} K)_{,y} - (e^{-\phi} K)(e^{-\phi} K^\mu{}_\nu) \\ &\quad + \frac{1}{2} \delta^\mu{}_\nu [(e^{-\phi} K)(e^{-\phi} K) + (e^{-\phi} K^{\alpha\beta})(e^{-\phi} K_{\alpha\beta})] - \nabla^\alpha \nabla_\alpha \phi - \nabla^\alpha \phi \nabla_\alpha \phi + \delta^\mu{}_\nu (\nabla^\alpha \nabla_\alpha \phi + \nabla^\alpha \phi \nabla_\alpha \phi) \\ &= \frac{(4+n)(3+n)}{2l^2} \delta^\mu{}_\nu + \kappa^2 (-\sigma^A \delta^\mu{}_\nu + T^A{}^\mu{}_\nu) e^{-\phi} \delta(y) + \kappa^2 (-\sigma^B \delta^\mu{}_\nu + \tilde{T}^B{}^\mu{}_\nu) e^{-\phi} \delta(y-l) , \end{aligned} \quad (5)$$

$${}^{(5+n)}G^y{}_y = -\frac{1}{2} R + \frac{1}{2} (e^{-\phi} K)(e^{-\phi} K) - \frac{1}{2} (e^{-\phi} K^{\alpha\beta})(e^{-\phi} K_{\alpha\beta}) = \frac{(4+n)(3+n)}{2l^2} , \quad (6)$$

$${}^{(5+n)}G^\mu{}_\mu = -\nabla_\nu (e^{-\phi} K^\mu{}_\nu) + \nabla_\mu (e^{-\phi} K) = 0 , \quad (7)$$

where $G^\mu{}_\nu = R^\mu{}_\nu - \delta^\mu{}_\nu R/2$ is the $(4+n)$ -dimensional Einstein tensor and ∇_μ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$. $T^\mu{}_\nu$ is the energy momentum

tensor of the brane matter other than the tension. The junction conditions are obtained by collecting together the terms in field equations which contain a δ -function,

then we obtain

$$e^{-\phi} [K_\nu^\mu - \delta_\nu^\mu K] |_{y=0} = \frac{\kappa^2}{2} (-\sigma_A \delta_\nu^\mu + T^{A\mu}_\nu), \quad (8)$$

$$e^{-\phi} [K_\nu^\mu - \delta_\nu^\mu K] |_{y=l} = -\frac{\kappa^2}{2} (-\sigma_B \delta_\nu^\mu + \tilde{T}^{B\mu}_\nu), \quad (9)$$

where $K_\nu^\mu = g^{\mu\alpha} K_{\alpha\nu}$. Note that the junction conditions constrain the induced metrics on both branes, they naturally give rise to the effective equations of motion for the gravity on the branes. In order to solve the bulk field equations, we use the gradient expansion scheme. The basic idea of the approximation is the assumption that the energy density of matter ρ on the brane is smaller than the brane tension σ . Equivalently, the bulk curvature scale l is much smaller than the characteristic length scale of the curvature L on the brane. Then, the small expansion parameter is given by $\epsilon = (l/L)^2 \ll 1$. This allows us to expand the metric in perturbative series starting from the induced metric on the A -brane $h_{\mu\nu}$ as the first term

$$g_{\mu\nu}(y, x^\mu) = a^2(y) \left[h_{\mu\nu}(x^\mu) + {}^{(1)}g_{\mu\nu}(y, x^\mu) + \dots \right], \quad (10)$$

where the boundary conditions on the A -brane are given by

$${}^{(i)}g_{\mu\nu}(y=0, x^\mu) = \begin{cases} h_{\mu\nu}(x^\mu) & : i=0, \\ 0 & : i=1, 2, 3, \dots \end{cases} \quad (11)$$

For the extrinsic curvature tensor we expand it as

$$K_\nu^\mu = {}^{(0)}K_\nu^\mu + {}^{(1)}K_\nu^\mu + {}^{(2)}K_\nu^\mu + \dots, \quad (12)$$

where ${}^{(i)}K_\nu^\mu = \mathcal{O}(\epsilon^i)$.

Applying the above scheme (see Appendix A for more detailed), we write down the $(4+n)$ -dimensional effective Einstein equations on the branes in closed form, subject to the low energy expansion as follows

$$G_\nu^\mu(h) = \frac{(2+n)\kappa^2}{2l} T^{A\mu}_\nu - \frac{(2+n)}{l} \chi_\nu^\mu, \quad (13)$$

$$G_\nu^\mu(f) = -\frac{(2+n)\kappa^2}{2l} T^{B\mu}_\nu - \frac{(2+n)}{l} \frac{\chi_\nu^\mu}{\Omega^{4+n}}, \quad (14)$$

where the A -brane metric is defined as $h_{\mu\nu} \equiv g_{\mu\nu}^{A-brane}$, while the B -brane metric is $f_{\mu\nu} \equiv g_{\mu\nu}^{B-brane}$. A conformal factor Ω relates the metric on the A -brane to that on the B -brane, $g_{\mu\nu}^{B-brane} = \Omega^2 g_{\mu\nu}^{A-brane}$. The terms proportional to χ_ν^μ are $(5+n)$ -dimensional Weyl tensor contributions, which describe the non-local $(5+n)$ -dimensional effect.

A. Effective theory on A -brane

Eliminating χ_ν^μ from equations (13) and (14), the $(4+n)$ -dimensional field equations on the A -brane can

be written as

$$G_\nu^\mu(h) = \frac{(2+n)\kappa^2}{2l} \frac{1}{\Psi} [T^{A\mu}_\nu + (1-\Psi)T^{B\mu}_\nu] + \frac{1}{\Psi} (\Psi|_\nu^\mu - \delta_\nu^\mu \Psi|_\alpha^\alpha) + \frac{\omega_A}{\Psi^2} \left(\Psi|^\mu \Psi|_\nu - \frac{1}{2} \delta_\nu^\mu \Psi|^\alpha \Psi|_\alpha \right), \quad (15)$$

where $|$ denotes the covariant derivative with respect to the A -brane metric $h_{\mu\nu}$ and the new scalar field $\Psi = 1 - \Omega^{2+n}$. The coupling function ω_A is defined as

$$\omega_A(\Psi) \equiv \frac{3+n}{2+n} \frac{\Psi}{1-\Psi}. \quad (16)$$

We can also determine χ_ν^μ by eliminating G_ν^μ from equations (13) and (14). Then, we obtain

$$\frac{(2+n)}{l} \chi_\nu^\mu = -\frac{(2+n)\kappa^2}{2l} \frac{(1-\Psi)}{\Psi} (T^{A\mu}_\nu + T^{B\mu}_\nu) - \frac{1}{\Psi} (\Psi|_\nu^\mu - \delta_\nu^\mu \Psi|_\alpha^\alpha) + \frac{\omega_A}{\Psi^2} \left(\Psi|^\mu \Psi|_\nu - \frac{1}{2} \delta_\nu^\mu \Psi|^\alpha \Psi|_\alpha \right). \quad (17)$$

Note that χ_ν^μ is expressed through the quantities on the branes, $\chi_\nu^\mu = \chi_\nu^\mu(x^\mu)$. Since χ_ν^μ is traceless, equation (17) leads to an equation of motion for the scalar field Ψ ,

$$\Psi|_\mu^\mu = \frac{1}{(3+n) + (2+n)\omega_A} \left[\frac{(2+n)\kappa^2}{2l} (T^A + T^B) - \frac{d\omega_A}{d\Psi} \Psi|^\mu \Psi|_\mu \right], \quad (18)$$

where we have taken Eq. (16) into account. The conservation laws for A -brane and B -brane matter with respect to the A -brane metric $h_{\mu\nu}$ are given by

$$T^{A\mu}_{\nu|\mu} = 0, \quad T^{B\mu}_{\nu|\mu} = \frac{\Psi|_\mu}{1-\Psi} T^{B\mu}_\nu - \frac{1}{(2+n)} \frac{\Psi|_\nu}{1-\Psi} T^B. \quad (19)$$

One can see that equations (15) and (19) do not include the term χ_ν^μ , but they include the energy momentum tensor of the B -brane. For this reason Kanno and Soda called this theory "quasi-scalar-tensor" gravity.

The effective action on A -brane can be derived from the original $(5+n)$ -dimensional action by substituting the solution of the equations of motion in the bulk and integrating out over the bulk coordinate. Up to the first order, we obtain the effective action for A -brane as,

$$S_A = \frac{l}{(2+n)\kappa^2} \int d^{4+n}x \sqrt{-h} \left[\Psi R(h) - \frac{\omega_A}{\Psi} \Psi|^\alpha \Psi|_\alpha \right] + \int d^{4+n}x \sqrt{-h} \mathcal{L}^A + \int d^{4+n}x \sqrt{-h} (1-\Psi)^{\frac{4+n}{2+n}} \mathcal{L}^B \quad (20)$$

Notice that the action (20) represents the action of the general $(4+n)$ -dimensional scalar-tensor theory with a specific form of the coupling function (16) and an extra matter term from the B -brane.

B. Effective theory on B -brane

To obtain the effective equations of motion on the B -brane, we simply reverse the role of the A -brane and that of the B -brane. Solving equation (14) for $G^\mu_\nu(f)$, the $(4+n)$ -dimensional field equations on the B -brane can be written as

$$\begin{aligned} G^\mu_\nu(f) = & \frac{(2+n)\kappa^2}{2l} \frac{1}{\Phi} [T^{B\mu}_\nu + (1+\Phi)T^{A\mu}_\nu] \\ & + \frac{1}{\Phi} (\Phi^{;\mu}_{;\nu} - \delta^\mu_\nu \Phi^{;\alpha}_{;\alpha}) \\ & + \frac{\omega_B}{\Phi^2} \left(\Phi^{;\mu} \Phi_{;\nu} - \frac{1}{2} \delta^\mu_\nu \Phi^{;\alpha} \Phi_{;\alpha} \right), \end{aligned} \quad (21)$$

where $;$ denotes the covariant derivative with respect to the B -brane metric $f_{\mu\nu}$ and $\Phi = \Omega^{-(2+n)} - 1$. Here, the coupling function ω_B is defined as

$$\omega_B(\Phi) = -\frac{3+n}{2+n} \frac{\Phi}{1+\Phi}. \quad (22)$$

The equations of motion for the scalar field Φ becomes

$$\begin{aligned} \Phi^{;\mu}_{;\mu} = & \frac{1}{(3+n) + (2+n)\omega_B} \left[\frac{(2+n)\kappa^2}{2l} (T^A + T^B) \right. \\ & \left. - \frac{d\omega_B}{d\Phi} \Phi^{;\mu} \Phi_{;\mu} \right]. \end{aligned} \quad (23)$$

The conservation laws of the A -brane and B -brane matter with respect to the B -brane metric $f_{\mu\nu}$ are as follows

$$T^{A\mu}_{\nu;\mu} = \frac{\Phi^{;\mu}}{1+\Phi} T^{A\mu}_\nu - \frac{1}{(2+n)} \frac{\Phi^{;\nu}}{1+\Phi} T^A, \quad T^{B\mu}_{\nu;\mu} = 0. \quad (24)$$

Finally, the corresponding effective action for B -brane is

$$\begin{aligned} S_B = & \frac{l}{(2+n)\kappa^2} \int d^{4+n}x \sqrt{-f} \left[\Phi R(f) - \frac{\omega_B}{\Phi} \Phi^{;\alpha} \Phi_{;\alpha} \right] \\ & + \int d^{4+n}x \sqrt{-f} \mathcal{L}^B + \int d^{4+n}x \sqrt{-f} (1+\Phi)^{\frac{4+n}{2+n}} \mathcal{L}^{(25)} \end{aligned}$$

In the derivation of equations of motion above we first to know the dynamics on one brane. Then we know the gravity on the other branes. Therefore, the dynamics on both branes are not independent. The transformation rules for scalar radion and the metric in $(4+n)$ -dimensions are given by

$$\Phi = \frac{\Psi}{1-\Psi}, \quad (26)$$

$$\begin{aligned} g_{\mu\nu}^{B\text{-brane}} = & (1-\Psi)^{\frac{2}{2+n}} \times \\ & \times \left[h_{\mu\nu} + g_{\mu\nu}^{(1)} (h_{\mu\nu}, \Psi, T_{\mu\nu}^A, T_{\mu\nu}^B, y=l) \right] \end{aligned} \quad (27)$$

The bulk metric is determined if we know the energy momentum tensors on both branes, the induced metric on A -brane, and the scalar field Ψ . Since $(4+n)$ -dimensional fields allow us to construct the $(5+n)$ -dimensional bulk geometry, the quasi-scalar-tensor theory works as a holographic at low energy.

In the following section, for the realization at the first order expansion, we study the cosmological consequences of the model. We solve the effective equations without knowing the bulk geometry. Then, we can determine the Friedman equation on the brane. Here we focus on the positive tension brane, A -brane.

III. KALUZA-KLEIN TWO BRANE WORLDS COSMOLOGY AT LOW ENERGY

A. Effective Friedmann equation

In this section, we discuss the cosmological consequences of the higher-dimensional brane worlds. We take the induced metric on A -brane of the form

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j + b^2(t) \delta_{\alpha\beta} dz^\alpha dz^\beta, \quad (28)$$

where δ_{ij} represents the metric of three-dimensional ordinary spaces with the spatial coordinates x^i ($i = 1, 2, 3$), while $\delta_{\alpha\beta}$ represents the metric of n -dimensional compact spaces with the coordinates z^α ($\alpha = 1, \dots, n$). The scale factor b denotes the size of the internal dimensions, while the scale factor a is the usual scale factor for the external space. We choose the energy momentum tensors of the A -brane and B -brane of the following form

$$T_{\mu\nu}^A = (\rho_A, P_A a^2 \delta_{ij}, Q_A b^2 \delta_{\alpha\beta}), \quad (29)$$

$$T_{\mu\nu}^B = \Omega^2 (\rho_B, P_B a^2 \delta_{ij}, Q_B b^2 \delta_{\alpha\beta}), \quad (30)$$

where ρ_i is the energy density, P_i the external pressure and Q_i the internal pressure, $i = A, B$. The Ω^2 factor results from the fact that the B -brane metric is $f_{\mu\nu} = \Omega^2 h_{\mu\nu}$. The symmetries imply that Ψ only depends on time.

Using the metric (28) and the energy momentum tensors (29), (30) in the effective Einstein equations (15), one finds

$$3H_a^2 + 3nH_aH_b + \frac{n(n-1)}{2}H_b^2 = \frac{8\pi G}{\Psi} \left[\rho_A + \rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right] + \frac{1}{\Psi} \left[\frac{(n+3)}{2(n+2)} \frac{\dot{\Psi}^2}{(1-\Psi)} - 3H_a\dot{\Psi} - nH_b\dot{\Psi} \right], \quad (31)$$

$$\begin{aligned} -2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 &= \frac{8\pi G}{\Psi} \left[P_A + P_B(1-\Psi)^{\frac{4+n}{2+n}} \right] \\ &+ \frac{1}{\Psi} \left[\ddot{\Psi} + \frac{(n+3)}{2(n+2)} \frac{\dot{\Psi}^2}{(1-\Psi)} + 2H_a\dot{\Psi} + nH_b\dot{\Psi} \right], \end{aligned} \quad (32)$$

$$\begin{aligned} -3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 &= \frac{8\pi G}{\Psi} \left[Q_A + Q_B(1-\Psi)^{\frac{4+n}{2+n}} \right] \\ &+ \frac{1}{\Psi} \left[\ddot{\Psi} + \frac{(n+3)}{2(n+2)} \frac{\dot{\Psi}^2}{(1-\Psi)} + 3H_a\dot{\Psi} + (n-1)H_b\dot{\Psi} \right], \end{aligned} \quad (33)$$

where we have defined the Hubble parameters $H_a = \dot{a}/a$ and $H_b = \dot{b}/b$ and

$$8\pi G = \frac{(2+n)\kappa^2}{2l}. \quad (34)$$

In the case $n = 0$, the above equations reduce to five-dimensional brane world. For $n = 0$, $\Psi = 1$, $\dot{\Psi} = 0$, the above equations reduce to the general relativistic FLRW equations with barotropic perfect fluid.

The equation of motion for the scalar field Ψ is

$$\begin{aligned} \ddot{\Psi} &= \frac{8\pi G}{(3+n)} [(\rho_A - 3P_A - nQ_A)(1-\Psi) \\ &+ (\rho_B - 3P_B - nQ_B)(1-\Psi)^{\frac{4+n}{2+n}}] \\ &- \frac{1}{2} \frac{\dot{\Psi}^2}{(1-\Psi)} - 3H_a\dot{\Psi} - nH_b\dot{\Psi}. \end{aligned} \quad (35)$$

In addition, the conservation laws for the matter with respect to the A -brane metric (19) are given by

$$\dot{\rho}_A + 3H_a(\rho_A + P_A) + nH_b(\rho_A + Q_A) = 0, \quad (36)$$

$$\begin{aligned} \dot{\rho}_B + 3H_a(\rho_B + P_B) + nH_b(\rho_B + Q_B) &= \\ \frac{3(\rho_B + P_B) + n(\rho_B + Q_B)}{2+n} \frac{\dot{\Psi}}{1-\Psi}. \end{aligned} \quad (37)$$

Substituting equation (35) into equations (32) and (33), respectively, and assuming the matter distribution on the branes are given by the equations of state $P_i = w_i\rho_i$ and $Q_i = v_i\rho_i$ ($i = A, B$). Equations (32) and (33) reduce to

$$\begin{aligned} -2\dot{H}_a - 3H_a^2 - 2nH_aH_b - n\dot{H}_b - \frac{n(n+1)}{2}H_b^2 + H_a\frac{\dot{\Psi}}{\Psi} &= \frac{8\pi G}{\Psi} \left[w_A\rho_A + \frac{(1-3w_A-nv_A)}{(3+n)}\rho_A(1-\Psi) \right. \\ &\left. + \frac{(1+nw_B-nv_B)}{(3+n)}\rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right] + \frac{1}{2(n+2)} \frac{\dot{\Psi}^2}{\Psi(1-\Psi)}, \end{aligned} \quad (38)$$

$$\begin{aligned} -3\dot{H}_a - 6H_a^2 - 3(n-1)H_aH_b - (n-1)\dot{H}_b - \frac{n(n-1)}{2}H_b^2 + H_b\frac{\dot{\Psi}}{\Psi} &= \frac{8\pi G}{\Psi} \left[v_A\rho_A + \frac{(1-3w_A-nv_A)}{(3+n)}\rho_A(1-\Psi) \right. \\ &\left. + \frac{(1-3w_B+3v_B)}{(3+n)}\rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right] + \frac{1}{2(n+2)} \frac{\dot{\Psi}^2}{\Psi(1-\Psi)}. \end{aligned} \quad (39)$$

From equations (31), (38), and (39), we eliminate $\dot{\Psi}^2$ term to obtain

$$2\dot{H}_a + \frac{3(4+n)}{(3+n)}H_a^2 + \frac{n(9+2n)}{(3+n)}H_aH_b + n\dot{H}_b + \frac{n(n^2+5n+2)}{2(3+n)}H_b^2 - \frac{n}{(3+n)}(H_a - H_b)\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi} \left[\frac{(1-(3+n)w_A)\rho_A}{(3+n)} - \frac{(1-3w_A-nv_A)\rho_A(1-\Psi)}{(3+n)} - \frac{n(w_B-v_B)}{(3+n)}\rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right], \quad (40)$$

$$\dot{H}_a + 3H_a^2 + (n-3)H_aH_b - \dot{H}_b - nH_b^2 + (H_a - H_b)\frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi} \left[(w_A - v_A)\rho_A + (w_B - v_B)\rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right] \quad (41)$$

Combining equations (40) and (41) we get the dynamical equation for Hubble parameters in $(4+n)$ -dimensions,

$$\begin{aligned} \dot{H}_a + 2H_a^2 + nH_aH_b + \frac{n(1+n)}{6}H_b^2 + \frac{n}{3}\dot{H}_b \\ = \frac{8\pi G}{3} \frac{(1-3w_A-nv_A)}{(2+n)}\rho_A. \end{aligned} \quad (42)$$

The conservation laws reduce to

$$\dot{\rho}_A + 3H_a(1+w_A)\rho_A + nH_b(1+v_A)\rho_A = 0, \quad (43)$$

$$\begin{aligned} \dot{\rho}_B + 3H_a(1+w_B)\rho_B + nH_b(1+v_B)\rho_B \\ = \frac{[3(1+w_B) + n(1+v_B)]\rho_B}{2+n} \frac{\dot{\Psi}}{1-\Psi}. \end{aligned} \quad (44)$$

In general, equation (42) is a second order differential equation for scale factor $a(t)$ and $b(t)$. In the case 4-

dimensional braneworld ($n = 0$), equation (42) can be solved analytically, and this results in the Friedmann equation on the brane with the dark radiation term as an integration constant. In our case equation (42) cannot be integrated analytically and therefore, the usual form of the Friedmann equation on the brane cannot be extracted. In the following two subsections we consider two cases: static and non-static internal dimensions.

B. Friedmann equation with static internal dimensions

In the case of static internal extra dimensions, the dynamical of the A -brane is described by the following equations

$$H_a^2 + H_a\frac{\dot{\Psi}}{\Psi} - \frac{(n+3)}{6(n+2)}\frac{\dot{\Psi}^2}{\Psi(1-\Psi)} = \frac{8\pi G}{3\Psi} \left[\rho_A + \rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right], \quad (45)$$

$$\dot{H}_a + 2H_a^2 = \frac{8\pi G}{3} \frac{(1-3w_A-nv_A)}{(2+n)}\rho_A, \quad (46)$$

$$\ddot{\Psi} + 3H_a\dot{\Psi} + \frac{1}{2}\frac{\dot{\Psi}^2}{(1-\Psi)} = \frac{8\pi G}{(3+n)} \left[(1-3w_A-nv_A)\rho_A(1-\Psi) + (1-3w_B-nv_B)\rho_B(1-\Psi)^{\frac{4+n}{2+n}} \right]. \quad (47)$$

Here we have assumed that the compact dimensions are stabilized, $b(t) = 1$ [44]. We see that the above equations do not contain any additional term compared with five-dimensional brane world cosmology. However, the differences from the usual two brane models are concealed in the gravitational constant and also in the form of the constraint equation (45).

The conservation laws for the matter with respect to the A -brane metric reduce to

$$\dot{\rho}_A + 3H_a(1+w_A)\rho_A = 0, \quad (48)$$

$$\begin{aligned} \dot{\rho}_B + 3H_a(1+w_B)\rho_B = \\ \frac{3(1+w_B)\rho_B + n(1+v_B)\rho_B}{2+n} \frac{\dot{\Psi}}{1-\Psi}, \end{aligned} \quad (49)$$

and we obtain

$$\rho_A \propto a^{-3(1+w_A)}, \quad (50)$$

$$\rho_B \propto a^{-3(1+w_B)}(1-\Psi)^{\frac{3(1+w_B)+n(1+v_B)}{2+n}}, \quad (51)$$

A relation between the energy densities on both branes can be obtained by eliminating a ,

$$\rho_B \propto \rho_A^{\frac{(1+w_B)}{(1+w_A)}} (1-\Psi)^{\frac{3(1+w_B)+n(1+v_B)}{2+n}}, \quad (52)$$

In the case $w_A \neq 1/3$, leaving v_A as a free parameter and using the matter conservation equation (48) we can write (46) as

$$\frac{d}{dt} \left(a^4 H_a^2 - \frac{8\pi G}{3} \frac{2(1-3w_A-nv_A)}{(2+n)(1-3w_A)} a^4 \rho_A \right) = 0. \quad (53)$$

Then, we obtain an expression for the effective Hubble parameter on A -brane as

$$H_a^2 = \frac{8\pi G_{eff}}{3} \rho_A + \frac{\mathcal{C}}{a^4}, \quad (54)$$

where \mathcal{C} is an integration constant which can be interpreted as dark radiation. We have defined the effective gravitational constant

$$G_{eff} = \frac{2(1 - 3w_A - nv_A)}{(2+n)(1 - 3w_A)} G. \quad (55)$$

For $w_A < 1/3$, $nv_A < 1 - 3w_A$ and $w_A > 1/3$, $nv_A > 1 - 3w_A$, the effective gravitational constant becomes positive.

In the case of radiation dominated universe, $w_A = 1/3$, we have

$$\dot{H}_a + 2H_a^2 = -\frac{8\pi G nv_A}{3(2+n)} \rho_A, \quad (56)$$

Using the matter conservation equation, we can write equation (56) as

$$\frac{d}{dt} \left(a^4 H_a^2 + \frac{8\pi G}{3} \frac{2nv_A}{(2+n)} \log a \right) = 0, \quad (57)$$

and giving

$$H_a^2 = -\frac{8\pi G}{3} \frac{2nv_A \log a}{(2+n)} \rho_A + \frac{K}{a^4}, \quad (58)$$

where K is an integration constant which can be redefined as a sum of the initial value of radiative matter density and initial value of the dark radiation density \mathcal{C} . Then equation (58) becomes

$$H_a^2 = \frac{8\pi G}{3} \left(1 - \frac{2n}{(2+n)} v_A \log \frac{a}{a_*} \right) \rho_A + \frac{\mathcal{C}}{a^4}, \quad (59)$$

where a_* is a constant corresponding to the dark radiation component \mathcal{C} . Defining the effective gravitational constant

$$G_{eff} = \left[1 - \frac{2n}{(2+n)} v_A \log \frac{a}{a_*} \right] G, \quad (60)$$

then we have the effective Friedmann equation (54). As expected the expression for the effective Friedmann equation on A -brane coincide with the Kaluza-Klein brane world cosmology with one brane model in the low energy approximation where the term of quadratic energy density is neglected [44]. In contrast to the usual four-dimensional two-brane model, the effective gravitational constant depends on the equation of state and the external scale factor explicitly, and may becomes positive or negative.

C. Friedmann equation with non-static internal dimensions

Let us now consider the case of non-static internal dimensions, in which the brane world evolves with two scale factors. We take a simple relation between the scale factors on A -brane of the form

$$b(t) = a^\gamma(t), \quad (61)$$

where γ is a constant. For the internal scale factor $b(t)$ to be small compared to the external scale factor $a(t)$, the constant γ should be negative.

For non-static internal dimensions, the dynamical of A -brane is described by the following equations

$$\left[\frac{6(1+n\gamma) + n(n-1)\gamma^2}{2} \right] H_a^2 + (3+n\gamma) H_a \frac{\dot{\Psi}}{\Psi} = \frac{8\pi G}{\Psi} \left[\rho_A + \rho_B (1-\Psi)^{\frac{4+n}{2+n}} \right] + \frac{(n+3)}{2(n+2)} \frac{\dot{\Psi}^2}{\Psi(1-\Psi)}, \quad (62)$$

$$\dot{H}_a + \frac{6(2+n\gamma) + n(1+n)\gamma^2}{2(3+n\gamma)} H_a^2 = \frac{8\pi G(1-3w_A - nv_A)}{(2+n)(3+n\gamma)} \rho_A, \quad (63)$$

$$\ddot{\Psi} + (3+n\gamma) H_a \dot{\Psi} = \frac{8\pi G}{(3+n)} \left[(1-3w_A - nv_A) \rho_A (1-\Psi) + (1-3w_B - nv_B) \rho_B (1-\Psi)^{\frac{4+n}{2+n}} \right] - \frac{1}{2} \frac{\dot{\Psi}^2}{(1-\Psi)}, \quad (64)$$

The conservation laws become

$$\dot{\rho}_A + [3(1+w_A) + n\gamma(1+v_A)] H_a \rho_A = 0, \quad (65)$$

$$\begin{aligned} & \dot{\rho}_B + [3(1+w_B) + n\gamma(1+v_B)] H_a \rho_B \\ &= \frac{[3(1+w_B) + n(1+v_B)] \rho_B}{2+n} \frac{\dot{\Psi}}{1-\Psi}, \end{aligned} \quad (66)$$

Using the matter conservation equation (65),

$$\begin{aligned} & [4\beta - 3(1+w_A) - n\gamma(1+v_A)] H_a \rho_A \\ &= \dot{\rho} + 4\beta H_a \rho_A = \frac{1}{a^{4\beta}} \frac{d}{dt} (a^{4\beta} \rho_A), \end{aligned} \quad (67)$$

and so we can write equation (63) as

$$\frac{d}{dt} \left(a^{4\beta} H_a^2 - \frac{8\pi G_{eff}}{3} a^{4\beta} \rho_A \right) = 0, \quad (68)$$

where

$$\beta = \frac{6(2+n\gamma) + n(1+n)\gamma^2}{4(3+n\gamma)}. \quad (69)$$

Then the effective Friedmann equation for non-static internal dimensions on A -brane is given by

$$H_a^2 = \frac{8\pi G_{eff}}{3} \rho_A + \frac{C}{a^{4\beta}}, \quad (70)$$

where C is a constant of integration and we have defined the effective gravitational constant as follows

$$G_{eff} = \frac{6(1-3w_A-nv_A)G}{(2+n)(3+n\gamma)[4\beta-3(1+w_A)-n\gamma(1+v_A)]}. \quad (71)$$

Notice that for $n=3$ and non-static internal dimensions, the setup is symmetric under the exchange of internal and external pressures ($w_i \leftrightarrow v_i$), and $a(t) \leftrightarrow b(t)$.

The above results also include the well-known five dimensional brane world, corresponding to $n=0$ and for which $\beta=1$, $G_{eff}=G$. For $\gamma=0$ the above results reduce to the static internal dimensions. If $\gamma=1$, the scale factor $b(t)$ is related to $a(t)$ as $b(t)=a(t)$, we obtain the Friedmann equation of the generalized Randall-Sundrum model in $(5+n)$ dimensions describing a $(4+n)$ -dimensional universe.

$$H_a^2 = \frac{8\pi G_{eff}}{3} \rho_A + \frac{C}{a^{4+n}}, \quad (72)$$

where the effective gravitational constant is now given by

$$G_{eff} = \frac{6}{(2+n)(3+n)} G. \quad (73)$$

In the case $n=0$, the above Friedmann equation reduces to usual Friedmann equation on four-dimensional brane.

Leaving β as a free parameter, we can solve equation (69) for γ . We obtain

$$\gamma = -\frac{3-2\beta \pm \sqrt{\frac{4\beta(3+n\beta)-3(4+n)}{n}}}{1+n}. \quad (74)$$

The negative values of γ indicate that the internal scale factor $b(t)$ to be small compared to the external scale factor $a(t)$. Taking $\beta=1$ such that the second term of Friedmann equation (70) contributes the "dark" radiation, we have

$$\gamma = -\frac{2}{1+n}, \quad \text{or} \quad \gamma = 0, \quad (75)$$

where $\gamma=0$ corresponds to the static internal dimensions. Therefore, the "dark" radiation component in the Friedmann equation can be also realized in the Kaluza-Klein brane worlds with non-static internal dimensions.

D. Hubble parameters in conformal frames

The action on A -brane is written in the Jordan frame, for which the gravitational sector has a non-canonical form. We can, however, perform a conformal transformation to the Einstein frame: $\tilde{h}_{\mu\nu} = \Psi^{2/(2+n)} h_{\mu\nu}$. In the Einstein frame, the metric (28) is

$$\begin{aligned} d\tilde{s}^2 &= \tilde{h}_{\mu\nu} dx^\mu dx^\nu \\ &= \Psi^{\frac{2}{2+n}} [-dt^2 + a^2(t) \delta_{ij} dx^i dx^j + b^2(t) \delta_{\alpha\beta} dz^\alpha dz^\beta] \\ &= -d\tilde{t}^2 + \tilde{a}^2(\tilde{t}) \delta_{ij} dx^i dx^j + \tilde{b}^2(\tilde{t}) \delta_{\alpha\beta} dz^\alpha dz^\beta, \end{aligned} \quad (76)$$

and the Hubble parameters satisfy

$$\tilde{H}_a - \tilde{H}_b = \Psi^{-\frac{1}{2+n}} (H_a - H_b), \quad (77)$$

where $\tilde{H}_a = \tilde{a}^{-1}(d\tilde{a}/d\tilde{t})$ and $\tilde{H}_b = \tilde{b}^{-1}(d\tilde{b}/d\tilde{t})$. One can see that the static internal dimensions (in the Jordan frame) may become dynamics in the Einstein frame. In this case we have,

$$\tilde{H}_a - \tilde{H}_b = \Psi^{-\frac{1}{2+n}} H_a, \quad \tilde{H}_b = \frac{1}{(2+n)\Psi} \frac{d\Psi}{d\tilde{t}}. \quad (78)$$

In the case $b(t) = a^\gamma(t)$, we have

$$\tilde{H}_a - \tilde{H}_b = (1-\gamma) \Psi^{-\frac{1}{2+n}} H_a. \quad (79)$$

Dynamics of the Hubble parameters H_a and H_b in the Jordan frame are also dynamics in the Einstein frame.

IV. CONCLUSION

In this paper we have derived the low energy effective equations for the higher-dimensional two brane models by using gradient expansion approximation. As expected, the effective theory is described by the $(4+n)$ -dimensional quasi-scalar-tensor gravity with a specific coupling function. The presented effective equations can be used as the basic equations for the higher-dimensional two brane worlds cosmology, in which some spatial dimensions on the brane are Kaluza-Klein compactified.

We can see already from the Friedmann equations that the Kaluza-Klein brane world can be realized at low energies. Due to their complicated structure the field equations appearing in the theories are very difficult to solve analytically, we have restricted our discussions with the special cases: static internal dimensions and non-static internal dimensions where a relation between the external and internal scale factors is given by $b(t) = a^\gamma(t)$. In the static internal dimensions $\gamma=0$, our results coincide with the Kaluza-Klein brane world cosmology with one brane model in the low energy approximation where the term of quadratic energy density is neglected [44]. In the non-static internal dimensions, the induced Friedmann equation on the brane is modified in the effective gravitational constant and the term proportional to $a^{-4\beta}$.

Another important result of this work is the dynamics of the internal Hubble parameter in conformal frames. Both the static and non-static internal dimensions in the Jordan frame are always dynamics in the Einstein frame. However, the physical interpretation and equivalence of these two frames is a problem in the case of static internal dimensions in the Jordan frame. We plan to investigate the correspondence between the Jordan and the Einstein frame description, including the dynamical of scalar field.

Acknowledgments

We thank the referees for many helpful suggestions which improved the presentation of the paper.

Appendix A: Detailed calculations

Let us decompose the extrinsic curvature into the traceless part and the trace part

$$e^{-\phi} K_{\mu\nu} = \Sigma_{\mu\nu} + \frac{1}{4+n} g_{\mu\nu} Q, \quad Q = -e^{-\phi} \frac{\partial}{\partial y} \log \sqrt{-g}, \quad (\text{A1})$$

which allows us to write the field equations (5) - (7) in the bulk as follows

$$e^{-\phi} \Sigma_{\nu,y}^\mu - Q \Sigma_{\nu}^\mu = - \left[R_{\nu}^\mu - \frac{1}{4+n} \delta_{\nu}^\mu R - \nabla^\mu \nabla_{\nu} \phi - \nabla^\mu \phi \nabla_{\nu} \phi + \frac{1}{4+n} \delta_{\nu}^\mu (\nabla^\alpha \nabla_{\alpha} \phi + \nabla^\alpha \phi \nabla_{\alpha} \phi) \right], \quad (\text{A2})$$

$$\frac{3+n}{4+n} Q^2 - \Sigma_{\beta}^{\alpha} \Sigma_{\alpha}^{\beta} = [R] + \frac{(4+n)(3+n)}{l^2}, \quad (\text{A3})$$

$$e^{-\phi} Q_{,y} - \frac{1}{4+n} Q^2 - \Sigma^{\alpha\beta} \Sigma_{\alpha\beta} = \nabla^\alpha \nabla_{\alpha} \phi + \nabla^\alpha \phi \nabla_{\alpha} \phi - \frac{4+n}{l^2}, \quad (\text{A4})$$

$$\nabla_{\lambda} \Sigma_{\mu}^{\lambda} - \frac{3+n}{4+n} \nabla_{\mu} Q = 0. \quad (\text{A5})$$

The junction conditions determine the dynamics of the induced metric and provide the effective theory of gravity on the brane reduced to

$$\left[\Sigma_{\nu}^{\mu} - \frac{3}{4} \delta_{\nu}^{\mu} Q \right] \Big|_{y=0} = \frac{\kappa^2}{2} (-\sigma_A \delta_{\nu}^{\mu} + T^{A\mu}_{\nu}), \quad (\text{A6})$$

$$\left[\Sigma_{\nu}^{\mu} - \frac{3}{4} \delta_{\nu}^{\mu} Q \right] \Big|_{y=l} = -\frac{\kappa^2}{2} (-\sigma_B \delta_{\nu}^{\mu} + \tilde{T}^{B\mu}_{\nu}). \quad (\text{A7})$$

1. Zeroth order

At zeroth order, the gradient terms and matter on the brane can be ignored. We find

$${}^{(0)}\Sigma_{\nu}^{\mu} = 0, \quad {}^{(0)}Q = \frac{4+n}{l}. \quad (\text{A8})$$

The junction conditions (A6) and (A7) yield

$$\sigma_A = \frac{2(3+n)}{\kappa^2 l}, \quad \sigma_B = -\frac{2(3+n)}{\kappa^2 l}. \quad (\text{A9})$$

Using the definition of the extrinsic curvature, we get the zeroth order metric as

$$ds^2 = e^{2\phi(y,x)} dy^2 + a^2(y,x) h_{\mu\nu} dx^{\mu} dx^{\nu}, \quad (\text{A10})$$

$$a(y,x) = \exp \left[-\frac{1}{l} \int_0^y dy e^{\phi(y,x)} \right], \quad (\text{A11})$$

where the tensor $h_{\mu\nu}$ is the induced metric on A -brane. To proceed we will assume $\phi(y,x) \equiv \phi(x)$ thus $a(y,x) = \exp [-ye^{\phi(x)}/l]$.

2. First order

In the first order, the curvature term that has been ignored in the zeroth order calculation comes into play. Substituting the solutions at zeroth order, the field equations (A2) - (A5) can be written as follows

$$e^{-\phi(1)} \Sigma_{\nu,y}^{\mu} - \frac{4+n}{l} {}^{(1)}\Sigma_{\nu}^{\mu} = - \left[R_{\nu}^{\mu} - \frac{1}{4+n} \delta_{\nu}^{\mu} R - (\nabla^{\mu} \nabla_{\nu} \phi + \nabla^{\mu} \phi \nabla_{\nu} \phi) + \frac{1}{4+n} \delta_{\nu}^{\mu} (\nabla^{\alpha} \nabla_{\alpha} \phi + \nabla^{\alpha} \phi \nabla_{\alpha} \phi) \right]^{(1)}, \quad (\text{A12})$$

$$\frac{2(3+n)}{l} {}^{(1)}Q = [R]^{(1)}, \quad (\text{A13})$$

$$e^{-\phi(1)} Q_{,y} - \frac{2}{l} {}^{(1)}Q = [\nabla^{\alpha} \nabla_{\alpha} \phi + \nabla^{\alpha} \phi \nabla_{\alpha} \phi]^{(1)} \quad (\text{A14})$$

$$\nabla_{\lambda} {}^{(1)}\Sigma_{\mu}^{\lambda} - \frac{3+n}{4+n} \nabla_{\mu} {}^{(1)}Q = 0. \quad (\text{A15})$$

And the junction conditions are given by

$$\left[{}^{(1)}\Sigma_{\nu}^{\mu} - \frac{3}{4} \delta_{\nu}^{\mu} {}^{(1)}Q \right] \Big|_{y=0} = \frac{\kappa^2}{2} T^{A\mu}_{\nu}, \quad (\text{A16})$$

$$\left[{}^{(1)}\Sigma_{\nu}^{\mu} - \frac{3}{4} \delta_{\nu}^{\mu} {}^{(1)}Q \right] \Big|_{y=l} = -\frac{\kappa^2}{2} \tilde{T}^{B\mu}_{\nu}. \quad (\text{A17})$$

where the superscript (1) represents the order of the gradient expansion. Now one can express the Ricci tensor $[R_{\nu}^{\mu}(g)]^{(1)}$ in term of the Ricci tensor of the A -brane

metric $h_{\mu\nu} \equiv g_{\mu\nu}^{A-brane}$ (denoted by $R^\mu_\nu(h)$) and ϕ ;

$$\begin{aligned} [R^\mu_\nu(g)]^{(1)} = & \frac{1}{a^2} \left[R^\mu_\nu(h) + \frac{(2+n)ye^\phi}{l} \left(\phi^\mu_{|\nu} + \phi^\mu_{|\alpha} \phi_{|\nu} \right) \right. \\ & + \frac{ye^\phi}{l} \delta^\mu_\nu \left(\phi^\alpha_{|\alpha} + \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & + \frac{(2+n)y^2e^{2\phi}}{l^2} \phi^\mu_{|\nu} \\ & \left. - \frac{(2+n)y^2e^{2\phi}}{l^2} \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right], \end{aligned} \quad (A18)$$

where $|$ denotes the covariant derivative with respect to the A -brane metric $h_{\mu\nu}$. Taking trace of equation (A18) and using equation (A13), the trace part of the extrinsic curvature can be obtained without solving the bulk geometry,

$$\begin{aligned} {}^{(1)}Q(y, x) = & \frac{l}{2(3+n)a^2} [R(g)] \\ = & \frac{l}{a^2} \left[\frac{1}{2(3+n)} R(h) + \frac{ye^\phi}{l} \left(\phi^\alpha_{|\alpha} + \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \right. \\ & \left. - \frac{(2+n)y^2e^{2\phi}}{2l^2} \phi^\alpha_{|\alpha} \phi_{|\alpha} \right], \end{aligned} \quad (A19)$$

The second derivatives of ϕ are given by

$$[\nabla^\mu \nabla_\nu \phi]^{(1)} = \frac{1}{a^2} \left[\phi^\mu_{|\nu} + 2 \frac{ye^\phi}{l} \phi^\mu_{|\nu} - \frac{ye^\phi}{l} \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right]. \quad (A20)$$

It is easy to see that the Hamiltonian constraint equation (A14) is trivially satisfied now. Then, equation (A12) can be integrated to give

$$\begin{aligned} {}^{(1)}\Sigma^\mu_\nu(y, x) = & \frac{l}{a^2} \left[\frac{1}{(2+n)} \left(R^\mu_\nu - \frac{1}{4+n} \delta^\mu_\nu R \right) \right. \\ & + \frac{ye^\phi}{l} \left(\phi^\mu_{|\nu} - \frac{1}{4+n} \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & + \left(\frac{y^2e^{2\phi}}{l^2} + \frac{ye^\phi}{l} \right) \times \\ & \times \left(\phi^\mu_{|\nu} - \frac{1}{4+n} \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & \left. + \frac{\chi^\mu_\nu(x)}{a^{4+n}} \right], \end{aligned} \quad (A21)$$

where $\chi^\mu_\nu(x)$ is an integration constant whose trace vanishes: $\chi^\mu_\mu = 0$, and equation (A15) requires that $\chi^\mu_{\nu|\mu} = 0$.

Substituting Eqs. (A19) and (A21) into the junction condition at the A -brane (A16), we obtain

$$\frac{l}{(2+n)} G^\mu_\nu(h) + \chi^\mu_\nu = \frac{\kappa^2}{2} T^{A\mu}_\nu, \quad (A22)$$

and the junction condition at the B -brane (A17) yields

$$\begin{aligned} & \frac{l}{(2+n)\Omega^2} G^\mu_\nu + \frac{le^\phi}{\Omega^2} \left(\phi^\mu_{|\nu} - \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & + \phi^\mu_{|\nu} - \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \\ & + \frac{le^{2\phi}}{\Omega^2} \left(\phi^\mu_{|\nu} + \frac{(1+n)}{2} \delta^\mu_\nu \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & + \frac{\chi^\mu_\nu}{\Omega^{4+n}} = -\frac{\kappa^2}{2\Omega^2} T^{B\mu}_\nu, \end{aligned} \quad (A23)$$

where $\Omega(x) = a(y = l, x) = \exp[-e^\phi]$ and the index of $T^{B\mu}_\nu$ is the energy momentum tensor with the index raised by the induced A -brane metric $h_{\mu\nu}$, while $\tilde{T}^{B\mu}_\nu$ is the one raised by the induced metric on the B -brane, $f_{\mu\nu} \equiv g_{\mu\nu}^{B-brane}$. Using $f_{\mu\nu} = \Omega^2 h_{\mu\nu} = \exp[-2e^\phi] h_{\mu\nu}$, equation (A23) can be rewritten as

$$\frac{l}{(2+n)} G^\mu_\nu(f) + \frac{\chi^\mu_\nu}{\Omega^{4+n}} = -\frac{\kappa^2}{2} \tilde{T}^{B\mu}_\nu, \quad (A24)$$

We now solve the metric in the bulk. The definition (A1) gives

$$-\frac{e^{-\phi}}{2a^2} h^{\alpha\mu} \frac{\partial}{\partial y} {}^{(1)}g_{\alpha\nu} = {}^{(1)}\Sigma^\mu_\nu + \frac{1}{4+n} \delta^\mu_\nu {}^{(1)}Q. \quad (A25)$$

Integrating Eq. (A25), we obtain the metric in the bulk:

$$\begin{aligned} {}^{(1)}g_{\mu\nu}(y, x) = & -\frac{l^2}{(2+n)} \left(\frac{1}{a^2} - 1 \right) \times \\ & \times \left[R_{\mu\nu} - \frac{1}{2(3+n)} h_{\mu\nu} R \right] \\ & + \frac{l^2}{2} \left(\frac{1}{a^2} - 1 - \frac{2ye^\phi}{l} \frac{1}{a^2} \right) \times \\ & \times \left(\phi_{|\mu\nu} + \frac{1}{2} h_{\mu\nu} \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & - \frac{y^2e^{2\phi}}{a^2} \left(\phi_{|\mu} \phi_{|\nu} - \frac{1}{2} h_{\mu\nu} \phi^\alpha_{|\alpha} \phi_{|\alpha} \right) \\ & - \frac{2l}{4+n} \left(\frac{1}{a^{4+n}} - 1 \right) \chi_{\mu\nu}, \end{aligned} \quad (A26)$$

where we have imposed the boundary condition, ${}^{(1)}g_{\mu\nu}(y = 0, x^\mu) = 0$. We can use a schematic iteration [13] for the solutions at higher orders.

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